RIGA TECHNICAL UNIVERSITY

Monte Carlo Methods in Financial Engineering

6. Topic

Brownian motion. Geometric Brownian motion.

Simulation of stochastic processes

- Methods for exact simulating continuous-time processes at discrete-time instances.
- The methods are considered precise in the sense that the joint distribution of simulated values matches the joint distribution of the continuous-time process, given the simulation time grid.

Brownian motion (one dimension)

- Standard one-dimensional Brownian motion on the interval $[0, T]$ – is a stochastic process $\{W(t), 0 \leq t \leq T\}$ with the following properties:
  - *i.* $W(0) = 0$;
  - ii. The representation $t \rightarrow W(t)$ is a continuous function on the interval $[0,\ T]$ with probability 1;
  - iii. The increments $\{W(t_1) - W(t_0), \mathrm{W}(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1})\}$ are independent for all k and for all $0 \leq\ t_0 < t_1 <\ \cdots <\ t_k \leq T$;
  - *iv.* $W(t) - W(s) \sim N(0, t - s)$ for each $0 \leq s < t \leq T$.

Brownian motion (one dimension)

- From (i) and (iv) follows that

$$W(t) \sim N(0, t), \qquad where\ 0 < t \leq T$$

- The process $X(t)$ is called a Brownian motion with a drift of μ and a diffusion coefficient σ^2 (constants μ and $\sigma > 0$), if

$$\frac{X(t) - \mu t}{\sigma}$$

is a standard Brownian motion.
It is denoted as $X \sim BM(\mu, \sigma^2)$.

Brownian motion (one dimension)

- So the process X can be constructed from the standard Brownian motion W, using

$$X(t) = \mu t + \sigma W(t)$$

- From this, it follows that $X(t) \sim N(\mu t, \sigma^2 t)$. Moreover, X is a solution to a stochastic differential equation (SDE)

$$dX(t) = \mu dt + \sigma dW(t)$$

Brownian motion (one dimension)

- For deterministic, but time-varying $\mu(t)$ and $\sigma(t) > 0$, can define a Brownian motion with drift μ and diffusion coefficient σ^2 using the following stochastic differential equation (SDE):

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

this is,

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$$

with an arbitrary constant $X(0)$.

Brownian motion (one dimension)

- The process X has continuous sample paths and independent increments.
- Each increment $X(t) - X(s)$ is normally distributed with a mean

$$E[X(t) - X(s)] = \int_s^t \mu(u)du$$

and variance

$$Var[X(t) - X(s)] = Var\left[\int_s^t \sigma(u)dW(u)\right] = \int_s^t \sigma^2(u)du$$

Random walk construction

- To simulate Brownian motion, we mainly consider simulating values $\left(W(t_1), \ldots W(t_n)\right)$ or $\left(X(t_1), \ldots X(t_n)\right)$ at fixed time points $0 < t_1 < \cdots < t_n$.
- Simulating the values $W(t_i)$ or $\mathrm{X}(t_i)$ using their increments is straightforward because the increments of Brownian motion are independent and normally distributed.
- Assume that $Z_1, \ldots, Z_n$ are independent random variables distributed according to the standard normal distribution.

Random walk construction

1. For the standard Brownian motion, choose $t_0 = 0$ and $\mathrm{W}(0) = 0$.
   The subsequent values can be generated as follows:
   $$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}, \qquad i = 0, \dots, n-1$$
2. For $X \sim BM(\mu, \sigma^2)$ with constants μ, σ and given $X(0)$:
   $$X(t_{i+1}) = X(t_i) + \mu \cdot (t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i} Z_{i+1}, \qquad i = 0, \dots, n-1$$
3. If the coefficients depend on time, use the following recursion:
   $$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(s) ds + \sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(u) du}\, Z_{i+1}, \qquad i = 0, \dots, n-1$$

Random walk construction

- Those three methods (slide 9) are precise in the sense that the joint distribution of simulated values $\left(W(t_1), \ldots, W(t_n)\right)$ or $\left(X(t_1), \ldots, X(t_n)\right)$ coincides with the joint distribution of the corresponding Brownian motion at times $t_1, \ldots, t_n$
  $\Rightarrow$ but from that, nothing can be known about what happens between t_i
- Therefore, these methods (slide 9) are subject to discretization error when compared to the true Brownian motion if deterministically interpolate at other time points.

Random walk construction

- The vector $\left(W(t_1), \ldots, W(t_n)\right)$ is a linear transformation of the vector composed of increments
$$\left(W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})\right).$$
- The increments being independent and normally distributed, the “property of linear transformations” implies that the vector $\left(W(t_1), \ldots, W(t_n)\right)$ possesses a multidimensional normal distribution.
- Therefore, simulating $\left(W(t_1), \ldots, W(t_n)\right)$ is a specific case of a more general problem – generating a multidimensional normally distributed vector.

Random walk construction

- In order to apply any method for generating a multidimensional normally distributed vector, you first need to find the mean vector and the covariance matrix of the vector $\big(W(t_1), \dots, W(t_n)\big)$.
- For a standard Brownian motion, $E[W(t_i)] = 0$, therefore, the mean vectors equal to 0.
- For the covariance matrix: first, for all $0 < s < t < T$, using the independence of increments, find

$$Cov[W(s), W(t)] = Cov\big[W(s), W(s) + \big(W(t) - W(s)\big)\big] =$$
$$= Cov[W(s), W(s)] + Cov[W(s), W(t) - W(s)] = s + 0 = s$$

- The covariance matrix of the vector $\big(W(t_1), \dots, W(t_n)\big)$ is denoted by C; then

$$C_{ij} = \min\big(t_i, t_j\big).$$

Cholesky factorization

- As the vector $\left(W(t_1), \dots, W(t_n)\right)$ follows the distribution $N(0, C)$, this vector can be simulated as AZ, where $Z = (Z_1, \dots, Z_n)^T \sim N(0,1)$ and A satisfies the equation $AA^T = C$.
- The *Cholesky* method chooses A as the lower triangular matrix. For the matrix C the *Cholesky* factor is in the following form:

$$A = \begin{pmatrix} \sqrt{t_1} & 0 & \cdots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_n - t_{n-1}} \end{pmatrix}$$

- In this case, generating the vector $\left(W(t_1), \dots, W(t_n)\right)$ as AZ implies the recursion $W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}$, which represents a matrix-vector representation.

Cholesky factorization

- For a process $X \sim BM(\mu, \sigma^2)$ the i-th component of the mean value vector of the vector $\left(X(t_1), \ldots, X(t_n)\right)$ is μt_i and the covariance matrix $\sigma^2 C$.
- The *Cholesky* factor is σA, and it can be confirmed that the *Cholesky* method aligns with the increment recursion

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}$$

Brownian Bridge construction

- The first recursion (from slide 9) generates the vector $\left(W(t_1), \ldots, W(t_n)\right)$ from left to right.
- However, the value $W(t_i)$ can be generated in any direction, ensuring that at each step, the sample is created from the correct *conditional distribution* determined by the values generated previously.
- For instance, you can first generate the last value $W(t_n)$, then $W\left(t_{\lfloor n/2 \rfloor}\right)$ under the condition of the existing value of $W(t_n)$, and so forth, obtaining intermediate values.

Brownian Bridge construction

- This flexibility can be useful in variance reduction techniques and *low-discrepancy* methods.
- From the "*Conditioning Formula*" it follows that the conditioned distribution needed at each step is itself a normal distribution, enabling this conditional sampling.
- Imposing conditions on the endpoints of Brownian motion creates a *Brownian bridge*.
- Determining $W(t_n)$ and then filling in the intermediate values results in simulating a Brownian bridge from $0 = W(0)$ to $W(t_n)$.

Brownian Bridge construction

- If the next step generates $W\left(t_{\lfloor n/2 \rfloor}\right)$, filling in values between the moments $t_{\lfloor n/2 \rfloor}$ and t_n results in simulating a Browning bridge from $W\left(t_{\lfloor n/2 \rfloor}\right)$ to $W(t_n)$. This approach is known as constructing a Brownian bridge.
- To develop such a construction: assume $0 < u < s < t$ and decide to generate $W(s)$ conditioned on the fact that $W(u) = x$ and $W(t) = y$.
- Apply the "*conditioning formula*" to find the conditioned distribution of $W(s)$.

Brownian Bridge construction

- From slide 12, it follows that the unconditional distribution is given by

$$\begin{pmatrix} W(u) \\ W(s) \\ W(t) \end{pmatrix} \sim N\left(0, \begin{pmatrix} u & u & u \\ u & s & s \\ u & s & t \end{pmatrix}\right)$$

- From the “conditioning formula”, we obtain the distribution of the second component of the vector under the condition of the value of the first component. This formula is applied to find the distribution of $W(s)$ under the condition of $\big(W(u), W(t)\big)$.

Brownian Bridge construction

- Therefore, first interchange the rows of the vector, obtaining

$$\begin{pmatrix} W(s) \\ W(u) \\ W(t) \end{pmatrix} \sim N\left(0, \begin{pmatrix} s & u & s \\ u & u & u \\ s & u & t \end{pmatrix}\right)$$

- From the “conditioning formula”, it is found that given $(W(u) = x, W(t) = y)$, $W(s)$ is normally distributed with a mean

$$W[W(s)|W(u) = x, W(t) = y] = \frac{(t-s)x + (s-u)y}{(t-u)}$$

and variance $\frac{(s-u)(t-s)}{(t-u)}$.

- The conditional mean is obtained by linearly interpolating between (u, x) and (t, y).

Brownian Bridge construction

- In a more general case, let's assume that the Brownian trajectory values $W(s_1) = x_1, W(s_2) = x_2, \dots, W(s_k) = x_k$ are specified at time moments $s_1 < s_2 < \cdots < s_k$, and we want to generate $W(s)$ under the condition for all these values. Assume that $s_i < s < s_{i+1}$. Then

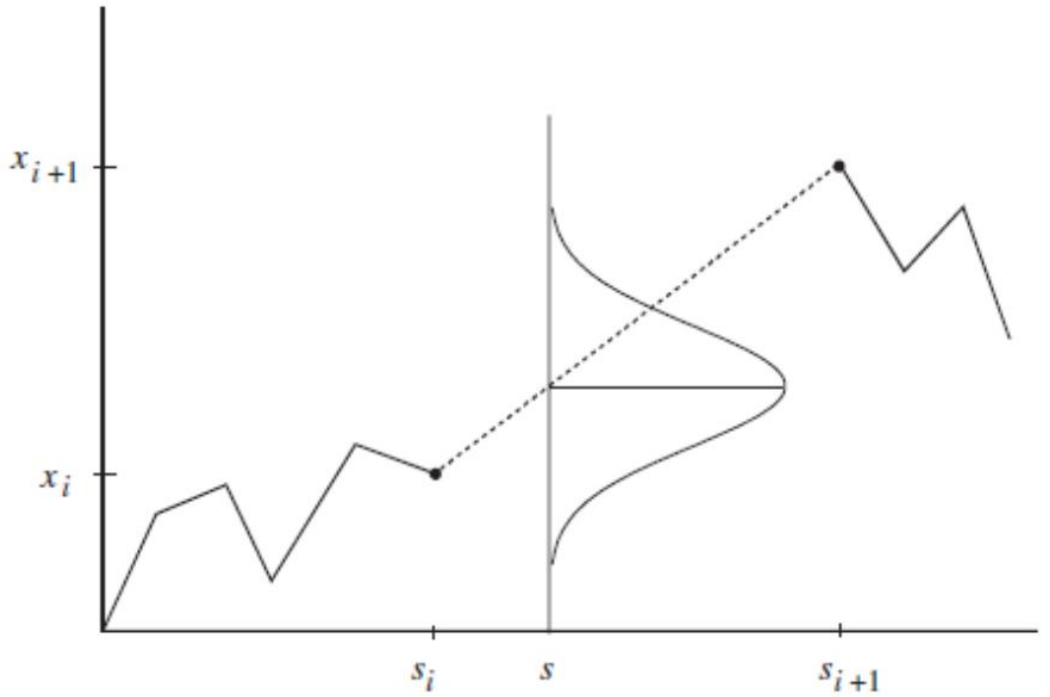
$$\left(W(s)\middle|W(s_j) = x_j, j = 1, \dots k\right) = (W(s)|W(s_i) = x_i, W(s_{i+1}) = x_{i+1})$$

- $(W(s)|W(s_1) = x_1, W(s_2) = x_2, \dots, W(s_k) = x_k) \sim$

$$N\left(\frac{(s_{i+1})x_i + (s - s_i)x_{i+1}}{(s_{i+1} - s_i)}, \frac{(s_{i+1} - s)(s - s_i)}{(s_{i+1} - s_i)}\right)$$

Brownian Bridge construction

- The conditional mean of $W(s)$ lies on the line segment connecting (s_i, x_i) and (s_{i+1}, x_{i+1}); the actual value of $W(s)$ is normally distributed around this mean with a variance that depends on $(s - s_i)$ and $(s_{i+1} - s)$



Brownian Bridge construction

- To generate such a conditional distribution, one can use

$$W(s) = \frac{(s_{i+1} - s)x_i + (s - s_i)x_{i+1}}{(s_{i+1} - s_i)} + \sqrt{\frac{(s_{i+1} - s)(s - s_i)}{(s_{i+1} - s_i)}}Z$$

With $Z \sim N(0,1)$ independent of all $W(s_1), \dots, W(s_k)$.

Reusing these observations, one can generate the components of vector $\left(W(t_1), \dots, W(t_n)\right)$ in any order. For instance, you can start by generating $W(t_n)$ from $N(0, t_n)$ and then proceed conditionally generating intermediate values, taking conditions on the two nearest time moments that have already been generated (possibly including $W(0) = 0$).

Implementation of Brownian bridge construction when the index is a number expressible as a power of 2.

Input: Time indices $(t_1, \ldots, t_{2^m})$
Output: Path $(w_1, \ldots, w_{2^m})$ with distribution of $(W(t_1), \ldots, W(t_{2^m}))$

Generate $(Z_1, \ldots, Z_{2^m}) \sim N(0, I)$
$h \leftarrow 2^m, \quad j_{\max} \leftarrow 1$
$w_h \leftarrow \sqrt{t_h} Z_h$
$t_0 \leftarrow 0, \quad w_0 \leftarrow 0$
for $k = 1, \ldots, m$
  $i_{\min} \leftarrow h/2, \quad i \leftarrow i_{\min}$
  $\ell \leftarrow 0, \quad r \leftarrow h$
  for $j = 1, \ldots, j_{\max}$
    $a \leftarrow ((t_r - t_i)w_\ell + (t_i - t_\ell)w_r)/(t_r - t_\ell)$
    $b \leftarrow \sqrt{(t_i - t_\ell)(t_r - t_i)/(t_r - t_\ell)}$
    $w_i \leftarrow a + bZ_i$
    $i \leftarrow i + h; \quad \ell \leftarrow \ell + h, \quad r \leftarrow r + h$
  end
  $j_{\max} \leftarrow 2 * j_{\max};$
  $h \leftarrow i_{\min};$
end
return $(w_1, \ldots, w_{2^m})$

Multidimensional Brownian motion

- The process $W(t) = \big(W_1(t), \dots, W_d(t)\big), 0 \le t \le T$, is called a standard Brownian motion in the space $\mathbb{R}^d$ if it satisfies that $W(0) = 0$, its trajectory is continuous, the increments are continuous, and
$$W(t) - W(s) \sim N(0, (t-s)I)$$
for all $0 \le s \le t \le T$, where I is $d \times d$ unity matrix.
- It follows that the process formed by each coordinate $W_i(t), i = 1, \dots, d$, is a one-dimensional standard Brownian motion process, and W_i and W_j are independent for $i \ne j$.

Multidimensional Brownian motion

- Assuming that μ is a vector in the space $\mathbb{R}^d$ and $\sum$ is $d \times d$ matrix that is positive definite or positive semi-definite, then the process X is called a Brownian motion with drift μ and covariance $\sum$ (denoted as $X \sim BM(\mu, \sum)$), if fulfilled, X has continuous trajectories and independent increments, where

$$X(t) - X(s) \sim N\big((t-s)\mu, (t-s)\sum\big)$$

- The initial value $X(0)$ is an arbitrary constant, which we'll consider as 0 unless specified otherwise.

Multidimensional Brownian motion

- If B is a $d \times k$ matrix where $BB^T = \sum$, and W is a standard Brownian motion in the space $\mathbb{R}^d$, then the
$$X(t) = \mu t + BW(t)$$
is $BM(\mu, \sum)$.
- Therefore, the matrix B influences the process X only through BB^T
- The process $X(t)$ is a solution to the following SDE:
$$dX(t) = \mu dt + BdW(t)$$

Multidimensional Brownian motion

- The definition of multidimensional Brownian motion can be extended to deterministic, time-varying $\mu(t)$ and $\sum(t)$ by using the equation

$$dX(t) = \mu(t)dt + B(t)dW(t)$$

  as the solution, where $B(t)B^T(t) = \sum(t)$.
- This process also has a continuous trajectory, independent increments, and

$$X(t) - X(s) \sim N\left(\int_s^t \mu(u)du\,, \int_s^t \sum(u)\ du\right)$$

Multidimensional Brownian motion

- If $X \sim BM(\mu, \sum)$, then
$$Cov\left[X_i(s), X_j(t)\right] = \min(s,t)\sum_{ij}$$
- Given $0 < t_1 < t_2 < \cdots < t_n$, can be easily found
$$\left(X_1(t_1), \ldots, X_d(t_1), X_1(t_2), \ldots, X_d(t_2), X_1(t_n), \ldots, X_d(t_n)\right)$$
the covariance matrix, by utilizing its mean value, thus reducing the problem of generating discrete trajectories of X to the generation of such an n-dimensional normal vector.
  - There might be situations where it is advantageous to apply this method, but generally, it is more convenient to use the fact that this n-dimensional vector is the concatenation of d vectors representing the process state at n different time points.

Random walk construction

- Assume $Z_1, Z_2, \ldots$ are independent $N(0, I)$ random vectors in the space $\mathbb{R}^d$. A standard d-dimensional Brownian motion can be constructed at times $0 = t_0 < t_1 < \cdots < t_n$, using $W(0) = 0$ and

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}, \qquad i = 0, \ldots, n-1$$

  That is equivalent to applying one-dimensional random walk construction separately to each coordinate of W.

Geometric Brownian motion

- Stochastic process $S(t)$ is referred to as a Geometric Brownian Motion (GBM) if $lnS(t)$ is a Brownian motion with initial value $lnS(0)$.
  - In other words, a geometric Brownian motion is an exponential Brownian motion.
  - Therefore, all methods used for simulating Brownian motions, when exponentiated, also become methods for simulating geometric Brownian motions.
- Geometric Brownian motion is the most fundamental model for evaluating financial assets.
  - Mainly attributed to the work of Paul Samuelson in the 1960s.

Brownian motion *vs* geometric Brownian motion

- Usually, Brownian motion can assume negative values - a feature undesired in stock price modeling – geometric Brownian motion is always positive as the exponential function only assumes positive values.
- Percentage change in geometric Brownian motion

$$\frac{S(t_2)-S(t_1)}{S(t_1)}, \frac{S(t_3)-S(t_2)}{S(t_2)}, \dots, \frac{S(t_n)-S(t_{n-1})}{S(t_{n-1})}$$

  are independent of time moments $t_1 < t_2 < \cdots < t_n$, unlike absolute changes $S(t_{i+1}) - S(t_i)$.
- These characteristics explain why in stock price modeling, geometric Brownian motion is more suitable than regular Brownian motion

GBM / properties

- Assuming W is a standard Brownian motion and X satisfies

$$dX(t) = \mu dt + \sigma dW(t)$$

and that $X \sim BM(\mu, \sigma^2)$. If we substitute $S(t) = S(0)\exp\big(X(t)\big) \equiv f\big(X(t)\big)$, then by applying Ito's formula, we obtain

$$dS(t) = S(t)\left(\mu + \frac{1}{2}\sigma^2\right)dt + S(t)\sigma dW(t) \qquad (1)$$

- On the other hand, the geometric Brownian motion process is often expressed in the form of an SDE as follows:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) \qquad (2)$$

This expression represents the model of Brownian motion for *instantaneous returns* $dS(t)/S(t)$

GBM / properties

- The notation: $S \sim GBM(\mu, \sigma^2)$

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$$

μ - the drift parameter

σ - $S(t)$ the volatility parameter

$\sigma^2 S^2(t)$ - $S(t)$ the diffusion coefficient

GBM / properties

- If $S \sim GBM(\mu, \sigma^2)$ and the initial value of S is $S(0)$, then

$$S(t) = S(0) \exp\left(\left[\mu - \frac{1}{2}\sigma^2\right] t + \sigma W(t)\right)$$

- In general, if $u < t$, then

$$S(t) = S(u) \exp\left(\left[\mu - \frac{1}{2}\sigma^2\right] (t-u) + \sigma\big(W(t) - W(u)\big)\right)$$

GBM / properties

- Since the increments of W are independent and normally distributed, for simulating the values of S at time points $0 = t_0 < t_1 < \cdots < t_n$, a simple recursive procedure can be used:

$$S(t_{i+1}) = S(t_i) \exp\left(\left[\mu - \frac{1}{2}\sigma^2\right](t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}\right),$$

$i = 0,1, \ldots, n-1$, with independent, standard normal $Z_1, Z_2, \ldots, Z_n$.

This method is an exact method in the sense that its generated random variables $\left(S(t_1), \ldots, S(t_n)\right)$ possess a common distribution (joint distribution) with the process $S \sim GBM(\mu, \sigma^2)$ at time points $t_1, \ldots, t_n$. This method does not have discretization errors.

Lognormal distribution

- The notation used is $Y \sim LN(\mu, \sigma^2)$ when the random variable Y follows a distribution of $\exp(\mu + \sigma Z)$, $Z \sim N(0,1)$.
- This distribution can be expressed as

$$P(Y \leq y) = P\left(Z \leq \frac{\ln(y) - \mu}{\sigma}\right) = \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right)$$

and density as

$$\frac{1}{y\sigma}\phi\left(\frac{\ln(y) - \mu}{\sigma}\right)$$

Lognormal distribution

- Moments of a lognormal random variable can be calculated by applying the following fundamental identity to the moment-generating function of the standard normal distribution

$$E[e^{aZ}] = e^{\frac{1}{2}a^2}$$

- It follows that for $Y \sim LN(\mu, \sigma^2)$:

$$E[Y] = e^{\mu+\frac{1}{2}\sigma^2}, \qquad Var[Y] = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$$

The notation $Y \sim LN(\mu, \sigma^2)$ does not imply that μ and σ^2 are the mean and the variance of Y.

GBM / Lognormal distribution

- If

$$S \sim GBM(\mu, \sigma^2)$$

then

$$\frac{S(t)}{S(0)} \sim LN\left(\left[\mu - \frac{1}{2}\sigma^2\right]t, \sigma^2 t\right)$$

and

$$E[S(t)] = e^{\mu t} S(0)$$
$$Var[S(t)] = e^{2\mu} S^2\left(e^{\sigma^2 t} - 1\right)$$

GBM / Lognormal distribution

- In fact:

$$E[S(t)|S(\tau), 0 \leq \tau \leq u] = E[S(t)|S(u)] = e^{\mu(t-u)}S(u)$$

  $u < t$.

  The first equation is the Markov property (it follows from the fact that S is a one-to-one transformation of Brownian motion, and Brownian motion is a Markov process).
- This equation shows that μ acts as the mean increment coefficient of S.

Task (assignment 6)

1. Generate standard Brownian motion over the interval [0,5] with 1000 steps using random walks.
2. Stock price changes $S(t) \sim GBM(\mu, \sigma^2)$. Simulate the dynamics of the stock price over the time interval $[0, T]$ and graphically represent it. Given:

$$S(0) = 50\ EUR, \mu = 0.05, T = 1\ year, \sigma = 0.35, n = 300$$

3. ***Optional*.**
   Construct a Brownian bridge according to the provided algorithm.

Thank You!