

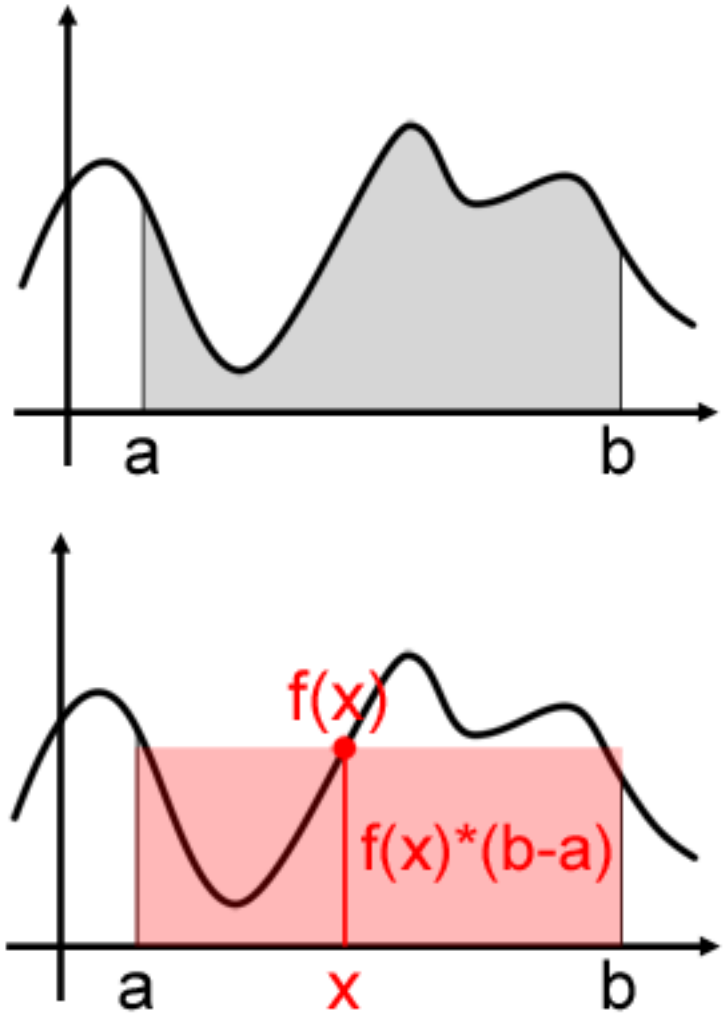

# Monte Carlo Methods in Financial Engineering

4. Topic

Integral Estimation

# Definite integral

$$F = \int_a^b f(x)dx$$



# Definite integral

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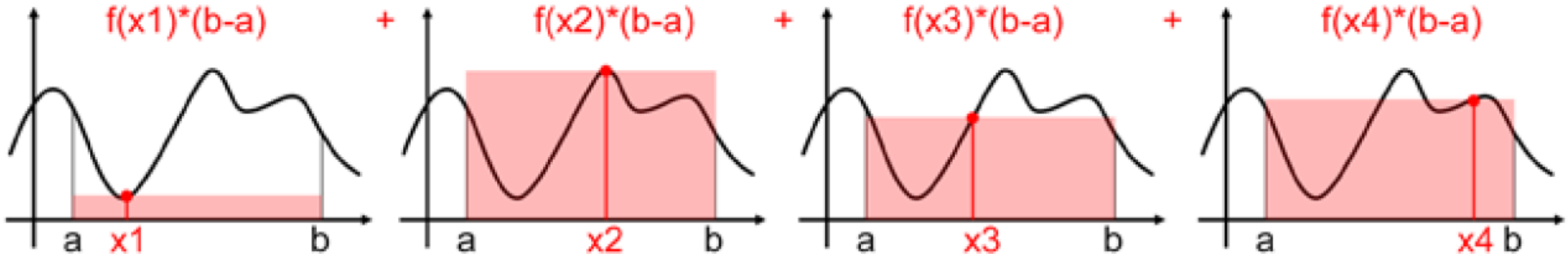




# Integral estimation

- Example – evaluation of the integral of the function $f$ over the interval [0;1]:

$$\propto = \int_0^1 f(x)dx$$

- Integral can be expressed as $E[f(U)]$, where $U \sim v.s.[0;1]$

$$E[f(U)] = \int_0^1 f(U)dU$$

- Generate $U_1, U_2, \ldots$

# Integral estimation

- By calculating the value of $f$ at these $n$ random points and taking the average, the result will be a Monte Carlo estimation

$$\hat{a}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i)$$

- If $f$ is integrable over the interval [0;1], then, according to the strict Law of Large Numbers

$\hat{a}_n \to a$ with probability of 1, when $n \to \infty$

# Integral estimation

- If $f$ is square integrable and we define $\sigma_f^2$ as

$$\sigma_f^2 \coloneqq \int_0^1 (f(x) - a)^2 dx$$

  So, the error of the Monte Carlo estimation $\hat{a}_n - a$ is approximately normally distributed with a mean of 0 and standard deviation of $\sigma_f / \sqrt{n}$.
- As the value of $n$ increases, the quality of the approximation improves.
- The parameter $\sigma_f$ is typically unknown if $a$ is unknown.

# Integral estimation

- However, this parameter $\sigma_f$ can be estimated using the standard deviation of the sample

$$s_f = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(f(U_i) - \hat{a}_n)^2}$$

- Thus, from the function values $f(U_1), \ldots, f(U_n)$ you obtain not only an estimate of the integral $a$, but also a measure of the error in this estimate.

# Integral estimation

- This form of standard error $\sigma_f / \sqrt{n}$ is an important indicator in the Monte Carlo method:
  - To reduce this error by half, it is necessary to increase the number of points by a factor of 4;
  - To increase the precision be one decimal place, 100 times more points are needed.
- In general, Monte Carlo is not a competitive method for calculating one-dimensional integrals.

# Integral estimation

- The advantage of the Monte Carlo method is that its convergence rate $O(n^{-\frac{1}{2}})$ is valid not only to integrals over the unit interval.
- All the previous considerations apply to integrals over the interval $[0,1]^d$ (and also to $\mathbb{R}^d$) for all dimensions $d$.
- Changing the dimensions ⇒ changes $f$ ⇒ changes $\sigma_f$, however, the standard error of the Monto Carlo estimate, which is based on $n$ points in $[0,1]^d$, will have the same form of $\sigma_f/\sqrt{n}$.
- The convergence rate $O\left(n^{-\frac{1}{2}}\right)$ remains consistent across all dimensions.

# Integral estimation

- Under certain conditions, it is useful to represent the price of derivative securities as the expected value. Thus, it reduces to the calculation of mathematical expectation.
- In many cases, if the corresponding mathematical expectations need to be written in integral form, this integral would be large or even infinite.

$\Longrightarrow$ This is the situation where the Monte Carlo method is most suitable.

# Integral estimation

- If you are integrating over the interval [a, b], then

$$\propto = \int_a^b f(x)dx\,, \quad \hat{a}_n = \frac{b-a}{n}\sum_{i=1}^{n} f(U_i)$$

- 2-dimensional case:

$$\propto = \int_a^b \int_c^d f(x,y)dxdy\,, \quad \hat{a}_n = \frac{(b-a)(d-c)}{n}\sum_{i=1}^{n} f(U_{1i}, U_{2i})$$

# Integral estimation

- d-dimensional case ($\mathbb{R}^d$):

$$\propto = \int_{[a,b]^d} f(x)dx = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_d}^{b_d} f(x_1,\cdots,x_d)dx_1\ \cdots dx_d$$

$$[a,b]^d = [a_1,b_1]\times\cdots\times[a_d,b_d]$$

# Tasks

1. Using the Monte Carlo method, calculate the integral and estimate the error. Compare it with the exact value of the integral.

$$\int_0^1 \frac{4}{1+x^2}\,dx$$

| $n$ | $\hat{a}_n$ | $s_f$ |
|---|---|---|
| 10 | | |
| 100 | | |
| 1000 | | |

# Tasks

2. $\int_0^4 \sqrt{x}dx$

3. $\int_0^1 \sqrt{x+\sqrt{x}}dx$

4. $\int_0^{5/4}(\int_0^{5/4}(4-x^2-y^2)\,dy)\,dx$

5. $\int_0^{5/4}(\int_0^{5/4}\sqrt{4-x^2-y^2}\,dy)\,dx$

6. $\int_0^{9/10}\left(\int_0^1\left(\int_0^{11/10}(4-x^2-y^2-z^2)dz\right)dy\right)dx$

# Thank You!