RIGA TECHNICAL UNIVERSITY

Monte Carlo Methods in Financial Engineering

2. Module

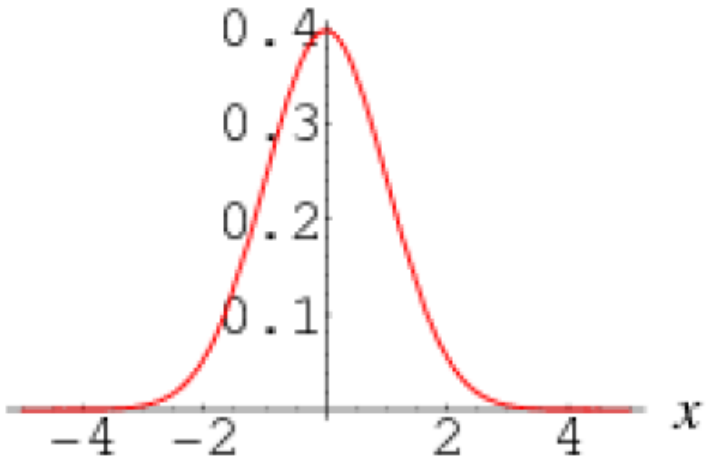
Normally Distributed Random Variables and Vectors Generation

Normal distribution

The standard normal distribution

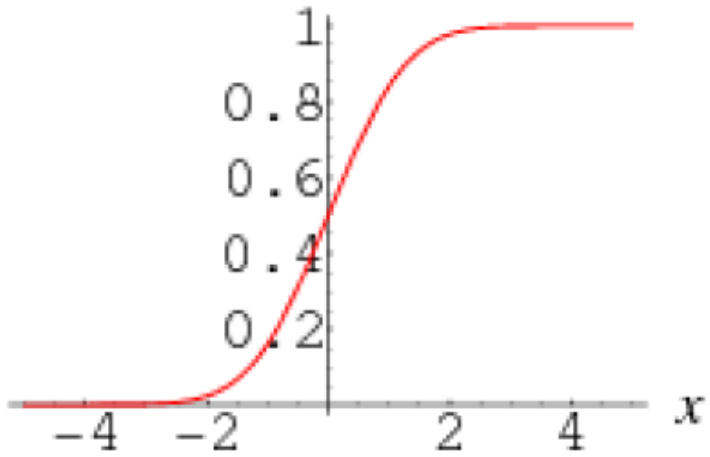
$X \sim N(0,1)$

Probability density function



$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

Cumulative distribution function



$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

Normal distribution

The standard normal distribution

With mean value μ and variance $\sigma^2, \sigma > 0$

$$X \sim N(\mu, \sigma^2)$$

- Probability density function

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

- Cumulative distribution function

$$\Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Normal distribution

If $Z \sim N(0,1)$ ⇒ $\mu + \sigma Z \sim N(\mu, \sigma^2)$

- The method for generating $Z_1, Z_2, \ldots$
  $\Longrightarrow$ The sample $X_1, X_2, \ldots$ with the distribution $N(\mu, \sigma^2)$ can be generated using the expression $X_i = \mu + \sigma Z_i$
  $\Longrightarrow$ So, it is sufficient to consider methods for generating samples with standard normal distribution $N(0,1)$

Normal distribution

Multidimensional normal distribution

d-dimensional normal distribution

$$X \sim N(\mu, \sum)$$

- μ is d-vector, $\sum$ is $d \times d$ covariance matrix;
- $\sum$ should be symmetric (i.e.,$\sum = \sum T$) and positive semi-definite

$$x^T \sum x \geq o, \qquad \forall x \in \mathbb{R}^d$$

- This means that all eigenvalues of $\sum$ must be non-negative.

Multidimensional normal distribution

- A special case, when $\sum$ is positive definite

$$x^T \sum x > 0, \qquad \forall x \in \mathbb{R}^d$$

(this means that all eigenvalues of $\sum$ are positive):

  - Probability density function ($|\sum|$ is the determinant of $\sum$)

$$\Phi_{\mu,\sum}(x) = \frac{1}{(2\pi)^{\frac{d}{2}} |\sum|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \sum^{-1} (x-\mu)\right), x \in \mathbb{R}^d$$

  - Probability density function for the standard normal distribution $X \sim N(0, I_d)$, where I_d is the $d \times d$ identity matrix

$$\phi(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{1}{2} x^T x\right)$$

Multidimensional normal distribution

- If $X \sim N(\mu, \sum)$, when
  - the i-th component $X_i \sim N(\mu_i, \sigma_i^2)$, where $\sigma_i^2 = \sum_{ii}$
  - the covariance between i-th and j-th components is:

    $$Cov[X_i, X_j] = E[(X_i - \mu_i)(X_j - \mu_j)] = \sum_{ij}$$

    $\sum$ is covariance matrix
  - Correlation between X_i and X_j : $\rho_{ij} = \frac{\sum_{ij}}{\sigma_i \sigma_j}$
  - Sometimes it is convenient to determine the marginal standard deviations $\sigma_i, i = 1, \ldots, d$, and correlations ρ_{ij}, from which covariance matrix elements can be expressed $\sum_{ij} = \sigma_i \sigma_j \rho_{ij}$

Normal distribution

- If the $d \times d$ symmetric matrix $\sum$ is positive semi-definite but not positive definite $\Longrightarrow$ the rank of $\sum$ is less than $d \Longrightarrow \sum$ does not have an inverse matrix $\Longrightarrow$ there is no corresponding normal density function for the covariance matrix $\sum$
  - In such a case, the normal distribution $N(\mu, \sum)$ can be defined as $X = \mu + AZ$, where $Z \sim N(0,, I_d) \forall d \times d$ matrix A, that satisfies $AA^T = \sum$
  - The resulting distribution does not depend on the choice of A
  - The random vector X does not have a probability density function in the $\mathbb{R}^d$ space, if the rank of $\sum$ is k, then you can find k components of X for which a multivariate normal density function exists in the $\mathbb{R}^k$ space.

Multivariate normal distribution properties

- **Linear transformation property**

Any linear transformation of a normally distributed vector is also normally distributed:

$$X \sim N(\mu, \sum) \Longrightarrow AX \sim N(A\mu, A\sum A^T)$$

for each d-dimensional vector μ and $d \times d$ matrix $\sum$, and each $k \times d$ matrix A for any k

Multivariate normal distribution properties

- **Conditioning formula**

Assume that the vector $(X_{[1]}, X_{[2]})$, where each $X_{[i]}$ can also be a vector, is multivariate normally distributed:

$$\begin{pmatrix} X_{[1]} \\ X_{[2]} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{[1]} \\ \mu_{[2]} \end{pmatrix}, \begin{pmatrix} \Sigma_{[11]} & \Sigma_{[12]} \\ \Sigma_{[21]} & \Sigma_{[22]} \end{pmatrix}\right)$$

And assume that $\Sigma_{[22]}$ has full rank. Then

$$(X_{[1]} | X_{[2]} = x) \sim N(\mu_{[1]} + \Sigma_{[12]}\Sigma_{[22]}^{-1}(x - \mu_{[2]}), \Sigma_{[11]} - \Sigma_{[12]}\Sigma_{[22]}^{-1}\Sigma_{[22]}\Sigma_{[21]})$$

This gives the distribution of $X_{[1]}$ with the condition of $X_{[2]} = x$.

Multivariate normal distribution properties

- **Moment generating function**

If X is d-dimensional vector with the distribution $X \sim N(\mu, \sum)$, then

$$E[\exp(\theta^T X)] = \exp\left(\mu^T \theta + \frac{1}{2}\theta^T \sum \theta\right)$$

For all $\theta \in \Re^d$.

Generating the standard normal distribution

- *Box-Muller* Method
- Modified *Box-Muller* Method

Box-Muller Method

- Generate a sample from a two-dimensional standard normal distribution such that each of its components follows a standard normal distribution

$$Z \sim N(0, I_2)$$

$$Z = (Z_1, Z_2)$$

$$Z_1, Z_2 \sim N(0,1)$$

Box-Muller Method

- The algorithm is based on the following two properties of the 2-dim standard normal distribution: IF $Z \sim N(0, I_2)$, then
  1. $R = {Z_1}^2 + {Z_2}^2$ follows an exponential distribution with a mean of 2, i.e.,
  $$P(R \leq x) = 1 - e^{-\frac{x}{2}}$$
  2. Given R, the point (Z_1, Z_2) is uniformly distributed on a circle with radius $\sqrt{R}$ and the center at the origin.
- To generate (Z_1, Z_2), first, generate R, and then uniformly select a point from the circle with a radius $\sqrt{R}$.

Box-Muller Method

Generate U_1, U_2 independent $Unif[0,1]$

$R \leftarrow -2\ln(U_1)$ - sample from an exponential distribution

$V \leftarrow 2\pi U_2$ - to generate a random point on a circle, you can generate uniformly distributed random angles between 0 and 2π

$Z_1 \leftarrow \sqrt{R}\cos(V), Z_2 \leftarrow \sqrt{R}\sin(V)$ - and then represent the angle as a point on the circumference of the circle; the coordinates give you the point on the circle's circumference

Obtain Z_1, Z_2

Modified Box-Muller Method

- Modified *Box-Muller* method reduces computation time by eliminating sine and cosine functions.
- The method uses the acceptance-rejection technique to obtain a set of points that are uniformly within a unit circle, and then transforms these points into normally distributed variables.

Modified Box-Muller Method

while $(X > 1)$

generate $U_1, U_2 \sim Unif[0,1]$

$U_1 \leftarrow 2 \cdot U_1 - 1, U_2 \leftarrow 2 \cdot U_2 - 1$- obtains (U_1, U_2) , which are uniformly distributed in the square $[-1,1] \times [-1,1]$

$\mathrm{X} \leftarrow {U_1}^2 {U_2}^2$- accepts only those pairs for which $X \leq 1$, obtains points that are uniformly distributed in the circle with a radius of 1 and center at the origin

end

$Y \leftarrow \sqrt{-2lnX/X}$ - dividing by $\sqrt{X}$, it projects the points from the unit circle onto a unit circle line, on which they are uniformly distributed.

$Z_1 \leftarrow U_1 Y, Z_2 \leftarrow U_2 Y$

Obtain Z_1, Z_2

Approximating the inverse distribution functions of the normal distribution

- Applying the inverse transform method to the normal distribution is associated with estimating Φ^{-1}
  - In the general case, this function cannot be calculated precisely
  - It cannot be approximated with sufficient accuracy in practical applications
- Since the normal distribution is symmetric, then

$$\Phi^{-1}(1-u) = -\Phi^{-1}(u), \qquad 0 < u < 1$$

This is sufficient to approximate Φ^{-1} in the interval [0.5, 1) (or in the interval (0,0.5]), and then use the symmetry property to extend the approximation to the remaining part of the unit interval.

Approximating the inverse distribution functions of the normal distribution

- *Beasley* and *Springer* propose a rational approximation

$$\Phi^{-1}(u) \approx \frac{\sum_{n=0}^{3} a_n \left(u - \frac{1}{2}\right)^{2n+1}}{1 + \sum_{n=0}^{3} b_n \left(u - \frac{1}{2}\right)^{2n}}, \qquad for\ 0.5 \leq u \leq 0.92$$

With given constants (Page 68, Figure 2.12);

For $u > 0.92$ is used a rational function

$$\sqrt{\ln(1-u)}$$

Approximating the inverse distribution functions of the normal distribution

- Moro also offers rational approximation (higher approximation ‘digits’ are obtained by replacing the second part of Beasley Springer’s approximation with Chebyshev approximation):

$$\Phi^{-1}(u) \approx g(u) = \sum_{n=0}^{8} c_n [\ln(-\ln(1-u))]^n, \qquad 0.92 \leq u < 1$$

With given constraints (Page 68, Figure 2.12, slide 21).

Appling the law of symmetry, you get

$$\Phi^{-1}(u) \approx -g(1-u), \qquad 0 < u \leq 0.08$$

Approximating the inverse distribution functions of the normal distribution

- Constants for the approximation of the normal distribution inverse function:

a_0 = 2.50662823884
a_1 = -18.61500062529
a_2 = 41.39119773534
a_3 = -25.44106049637

b_0 = -8.47351093090
b_1 = 23.08336743743
b_2 = -21.06224101826
b_3 = 3.13082909833

c_0 = 0.3374754822726147
c_1 = 0.976169019091718 6
c_2 = 0.160797971491820 9
c_3 = 0.027643881033386 3
c_4 = 0.003840572937360 9

c_5 = 0.000395189651191 9
c_6 = 0.000032176788176 8
c_7 = 0.000000288816736 4
c_8 = 0.000000396031518 7

Approximating the inverse distribution functions of the normal distribution

Input: u in the range of 0 to 1; Result: $x, \Phi^{-1}(u)$ approximation [Beasley-Springer, Moro]

$y \leftarrow u - 0.5$

$if\ |y| < 0.42$

$r \leftarrow y \cdot y$

$x \leftarrow y \cdot \Big(\big((a_3 \cdot r + a_2) \cdot r + a_1\big) \cdot r + a_0\Big) / \Big(\big((b_3 \cdot r + b_2) \cdot r + b_1\big) \cdot r + b_0 + 1\Big)$

$else$

$r \leftarrow u;$

$if (y > 0) r \leftarrow 1 - u$

$r \leftarrow \ln(-\ln(r))$

$$x \leftarrow c_0 + r\left(c_1 + r\left(c_2 + r\left(c_3 + r\left(c_4 + r\left(c_5 + r\big(c_6 + r(c_7 + rc_8)\big)\right)\right)\right)\right)\right)$$

$if\ (y < 0)\ x \leftarrow -x$

$return\ x$

Approximation of the cumulative function of the normal distribution

- The estimation of the cumulative distribution function of the normal distribution is required in many financial applications (including the evaluation of *Black-Scholes* formulas)
- Two methods:
  1. *Hastings* – faster method;
  2. *Marsaglia et al.* – more precise method

  Both methods are sufficiently fast and precise for the majority of application.

Approximation of the cumulative function of the normal distribution

1. Based on *Hastings* work

For $x \geq 0$, it has the form

$$\Phi(x) \approx 1 - \phi(x)(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5),$$

$$t = \frac{1}{1+px}$$

b_i and p are constants.

For negative arguments, the approximation is extended using the equality $\Phi(-x) = 1 - \Phi(x)$.

Approximation of the cumulative function of the normal distribution

1. Based on the work of *Hastings*
   (methods maximum absolute error is $< 7.5 \times 10^{-8}$)

Constants:

b_1 = 0.319381530
b_2 = -0.356563782
b_3 = 1.781477937
b_4 = -1.821255978
b_5 = 1.330274429

p = 0.2316419
$c = \ln(\sqrt{2\pi}) = 0.918938533204672$

Approximation of the cumulative function of the normal distribution

1. *Hastings'* approximation after modification by *Abramowitz* and *Stegun*

Input: x; Result: $y, \Phi(x)$ approximation

$a \leftarrow |x|$

$t \leftarrow 1/(1 + a \cdot p)$

$$s \leftarrow \Big(\Big(\big((b_5 \cdot t + b_4) \cdot t + b_3\big) \cdot t + b_2\Big) \cdot t + b_1\Big) \cdot t$$

$y \leftarrow s \cdot \exp(-0.5 \cdot x \cdot x - c)$

$if\ (x > 0)\ \ y \leftarrow 1 - y$

$return\ y$

Approximation of the cumulative function of the normal distribution

2. *Marsaglia et al.*

Based on this relationship approximation

$$\big(1-\Phi(x)\big)/\phi(x)$$

For this method, the maximum relative error is up to 10^{-15} for $0 \le x \le 6.23025$ and 10^{-12} for larger x.

Approximation of the cumulative function of the normal distribution

2. Input: x between -15 and 15; Result: $y, \Phi(x)$ approximation

$j \leftarrow \lfloor \min(|x| + 0.5, 14) \rfloor$

$z \leftarrow j, h \leftarrow |x| - z, a \leftarrow v_{j+1}$ - v_{j+1} is a constant as given in Figure 2.15 Page 70

$b \leftarrow z \cdot a - 1, q \leftarrow 1, s \leftarrow a + h \cdot b$

$for\ i = 2,4,6, \ldots, 24 - j$

$a \leftarrow (a + z \cdot b)/i$

$b \leftarrow (b + z \cdot a)/(i + 1)$

$q \leftarrow q \cdot h \cdot h$

$s \leftarrow s + q \cdot (a + h \cdot b)$

end

$y = s \cdot \exp(-0.5 \cdot x \cdot x - c)$

$if\ (x > 0)\ y \leftarrow 1 - y$

$return\ y$

Task

Task 4.

Generate a sample from a two-dimensional standard normal distribution with 500 elements using the *Box-Muller* method.

Thank You!