RIGA TECHNICAL UNIVERSITY

Monte Carlo Methods in Financial Engineering

8. Topic

The effectiveness of simulation evaluation.

The effectiveness of simulation evaluation

- Methods of improving Monte Carlo evaluation.
- 3 criteria that are important when considering improvements and comparing them to alternative evaluations:
  1. computation time,
  2. deviation/bias,
  3. variance
- Let's assume that

$$\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i \,,$$

where C_i, $E[C_i] = C$ and $Var[C_i] = e_C^2 < \infty$

The effectiveness of simulation evaluation

- The Central Limit Theorem states:
  - As the number of representations n increases, the standardized evaluation

$$\frac{\hat{C}_n - C}{\sigma_C / \sqrt{n}}$$

    in distribution converges to the standard normal distribution;
  - Represented as

$$\frac{\hat{C}_n - C}{\sigma_C / \sqrt{n}} \Rightarrow N(0,1)$$

    or equivalently

$$\sqrt{n}(\hat{C}_n - C) \Rightarrow N(0, \sigma_C^2)$$

The effectiveness of simulation evaluation

- Convergence in distribution implies that

$$\lim_{n\to\infty} P\left(\frac{\hat{C}_n - C}{\sigma_C/\sqrt{n}} \leq x\right) = \Phi(x)$$

  for all x, where Φ represents the cumulative normal distribution.
- This limit also holds true if σ_C is replaced by the sample standard deviation s_C
  - This is important because in practice, σ_C is rarely known, but s_C can be calculated from simulation results.

The effectiveness of simulation evaluation

- From the Central Limit Theorem, one can infer the distribution of the simulation estimation error as:

$$\hat{C}_n - C \approx N\left(0, \frac{\sigma_C^2}{n}\right)$$

  This means that the error $\hat{C}_n - C$ is approximately distributed according to a normal distribution.
  - This provides a basis for comparing two estimation of different magnitudes when all other indicators match, it's advisable to choose the estimation with lower variance.
  - But if the other indicators do not match?

The effectiveness of simulation evaluation

- For instance, when choosing between two unbiased estimations, one with lower variance but requiring a longer computation time.
  - How to balance reducing variance and the computational efforts/resources involved in calculations?
  - Authors such as *Hammersley* and *Handscomb*, *Fox* and *Glynn*, *Glynn* and *Whitt* developed general approaches to analyze this.
  - Let’s explore some key conclusion.

The effectiveness of simulation evaluation

τ – Computation time required for generating a single C_i iteration

s – Computational/budget constraints

(The objective is to compare estimations based on relative computational efforts, hence the units in which computation time is measured are not crucial. s is measured in the same units as τ).

Therefore, the number of iterations that can be performed within the available budget is $\lfloor s/\tau \rfloor$ (the integer part), and the resulting estimation is $\hat{C}_{\lfloor s/\tau \rfloor}$.
Obtained:

$$\sqrt{\lfloor s/\tau \rfloor}\left(\hat{C}_{\lfloor s/\tau \rfloor} - C\right) \Rightarrow N(0, \sigma_C^2)$$

if $s \to \infty$

The effectiveness of simulation evaluation

As

$$\frac{\lfloor s/\tau \rfloor}{s} \to \frac{1}{\tau},$$

it follows that

$$\sqrt{s}\left(\hat{C}_{\lfloor s/\tau \rfloor} - C\right) \Rightarrow N(0, \sigma_C^2 \tau),$$

as $s \to \infty$.

From this, it can be concluded that at a given budget s, the estimation error will be approximately distributed according to the law of normal distribution with variance $\sigma_C^2 \tau / s$.

The effectiveness of simulation evaluation

This characteristic provides criteria for comparing unbiased estimations.

For example:

Suppose there are two unbiased estimations, both as the mean value from independent iterations. Let σ_1^2 be the variance of each iteration for the first estimation, σ_2^2 for the second, and $\sigma_1^2 > \sigma_2^2$. However, the computation time for iteration $\tau_1 < \tau_2$. How should you choose?

According to the Central Limit Theorem, asymptotically (as the computational budget increases), preference should be given to the estimation with a smaller $\sigma_i^2 \tau_i$ value. This is because it will be a more precise estimation (and with a narrower confidence interval) within the budget s constraints.

The effectiveness of simulation evaluation

So far, it has been assumed that the computation time for each iteration is constant.

- This holds true when considering, for instance, Asian options and the pricing of many derivative securities, where the computation time for each iteration is initially determined by choosing the number of simulation time step.
- However, there are cases where the computation time significantly varies during iterations. For example, in barrier options, where the observation of the first time when the trajectory crosses a pre-determined level/barrier occurs. This transitions time is of a stochastic nature.

The effectiveness of simulation evaluation

To generalize conclusions in evaluating estimation efficiency, let's introduce the notation:

$$(C_1, \tau_1), (C_2, \tau_2), \ldots$$

where $(C_i, \tau_i) \sim i.i.d., \tau_i$ - represents the computation time for the i-th iteration.

The number of iterations that can be performed within the computation budget s Is:

$$N(s) = \sup\left\{n \geq 0: \sum_{i=1}^{n} \tau_i \leq s\right\},$$

and this has a stochastic nature.

The effectiveness of simulation evaluation

The estimation corresponding to the budget s is $\hat{C}_{N(s)}$, which is the average value from the first $N(s)$ iterations.

The $i.i.d.$ assumption ensures that with a probability of 1, it holds that

$$\frac{N(s)}{s} \to \frac{1}{E[\tau]}$$

and that

$$\sqrt{s}\left(\hat{C}_{N(s)} - C\right) \Rightarrow N(0, \sigma_C^2 E[\tau]).$$

This limit provides the opportunity to measure the asymptotic relative efficiency when the computation time for iterations varies.

The effectiveness of simulation evaluation

This means that when comparing estimations constructed as averages from unbiased independent iterations, the estimation with the smallest $\sigma_C^2 E[\tau]$ should be chosen.

There are even more general results where there's no assumption about $(C_i, \tau_i) \sim i.i.d.$

Bias

- The comparison of efficiency in previous slides was based on these estimations being defined as averages from unbiased iterations.
- When there is no bias, the utmost attention is directed towards the variability of the estimation and the computational resources consumed.
- In small samples, there are instances where accepting bias might be necessary, but primarily, the focus lies on those estimations where bias can be minimized by increasing computational efforts/resources.

Bias

- Sometimes, simulation estimation is biased when the iterations are of a finite count, but it becomes asymptotically unbiased as the number of iterations increases.
  - For example, in the case of $\hat{C}_{N(s)}$. If τ_i is a random quantity, $E[\hat{C}_{N(s)}] \neq C$, but the Central Limit Theorem shows that the bias in this case becomes insignificant as s increases.
  - Another example is when evaluating the mathematical expectation relationship $E[X]/E[Y]$ for an i.i.d. pair (X, Y) of iterations $(X_i, Y_i), i = 1, \dots, n$. The sample average ratio $\bar{X}/\bar{Y}$ is biased $\forall n$ because $E[\bar{X}/\bar{Y}] \neq E[\bar{X}]/E[\bar{Y}]$, but $\bar{X}/\bar{Y}$ converges to $E[X]/E[Y]$ with probability 1 as $n \to \infty$. However, the normalized error $\sqrt{n}(\bar{X}/\bar{Y} - E[X]/E[Y])$ is asymptotically normal. Thus, the bias becomes insignificant as the number of iterations increases.

Bias

- However, the bias doesn’t always automatically disappear in large samples; sometimes special attention is required.
- Here are three examples that demonstrate typical non-negligible biases in financial engineering simulations:
  1. Model discretization error;
  2. Payoff discretization error;
  3. Nonlinear functions of means.

Bias / 1 example

Model discretization error.

Applying the Monte Carlo method to estimate the expected present value of the payoff for a standard call option yields an unbiased estimation by simulating the underlying stock with

$$S(T) = S(0)\exp\left(\left[r - \frac{1}{2}\sigma^2\right]T + \sigma\sqrt{T}Z\right)$$

and (direct method)

$$S(t_{j+1}) = S(t_j)\exp\left(\left[r - \frac{1}{2}\sigma^2\right](t_{j+1} - t_j) + \sigma\sqrt{t_{j+1} - t_j}Z_{j+1}\right)$$

Bias / 1 example

Model discretization error.

Let's assume that instead of the previous equation, the time interval is divided into small intervals of length h, and the dynamics of the underlying stock are evaluated using recursion:

$$S\big((j+1)h\big) = S(jh) + rS(jh)h + \sigma S(jh)\sqrt{h}Z_{j+1}$$

where $Z_1, Z_2, \ldots$ are independent standard normally distributed random variables.

By using such a simulation relationship, the estimation of the expected present value of the option's payoff will differ from the true value - the estimation is biased.

Bias / 1 example

Model discretization error.

This is discretization bias because it arises from the time discretization in a model with continuous-time dynamics.

In many models with continuous-time dynamics, the direct method (slide 17) is not feasible, and discretization error is unavoidable.

- This typically occurs in cases where the volatility parameter σ is a function of stock prices S.

Bias / 1 example

Model discretization error.

Such discretization error is controllable because it typically diminishes (approaches zero) as the time step h decreases.

However, reducing h means more transitions within one trajectory (in the case of a fixed time interval), resulting in higher computational resources being consumed.

Bias / 2 example

Payoff discretization error.

Let's assume that for Asian options is used the continuous average value $\bar{S} = \frac{1}{T}\int_0^T S(u)du$.

If the $S(t_i)$ values are calculated at discrete time points, for example, using

$$S(t_{j+1}) = S(t_j)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma\sqrt{t_{j+1} - t_j}Z_{j+1}\right),$$

it's not possible to precisely calculate $\bar{S}$. A discrete approximation is required for the continuous average.

Bias / 2 example

Payoff discretization error.

A similar situation is also observed when evaluating the expected present value of the *payoff* for *lookback* options:

$$E\left[e^{-rT}\left(\max_{0\le t\le T} S(t) - S(t)\right)\right].$$

Even if the trajectories $S(0), S(t_1), \ldots, S(t_m)$ are simulated accurately, the estimation of $e^{-rT}\left(\max_{o\le j\le m} S(t_j) - S(T)\right)$ is biased.

Bias / 2 example

Payoff discretization error.

In both aforementioned cases, biases can be made arbitrarily small by using a sufficiently small time step in the simulation, albeit at the expense of increased computational costs for each trajectory.

In this scenario, discretization error arises in the form of the option's payoff; the values of $S(t_i)$ are modeled without discretization error.

Compared to model discretization errors, this type of bias is less common in practice.

Bias / 3 example

Nonlinear functions of the mean value.

Let's assume the option expires at T_1,
this option provides the opportunity to buy a call option with a term $T_2 > T_1$;

this is often referred to as a compound option or an option on an option.

Bias / 3 example

Nonlinear functions of the mean value.

$C^{(2)}(x)$ - the conditional expected discounted payoff of the option at time T_1 concerning the underlying stock price x for an option with a term T_2 is expressed as:

$$C^{(2)}(x) = E\left[e^{-r(T_2-T_1)}(S(T_2)-K_2)^+ \middle| S(T_1)=x\right]$$

If the strike price for the compound option is K_1, then the expected present value of its payoff is:

$$C^{(1)} = E\left[e^{-rT_1}\left(C^{(2)}\left(S(T_1)\right)-K_1\right)^+\right]$$

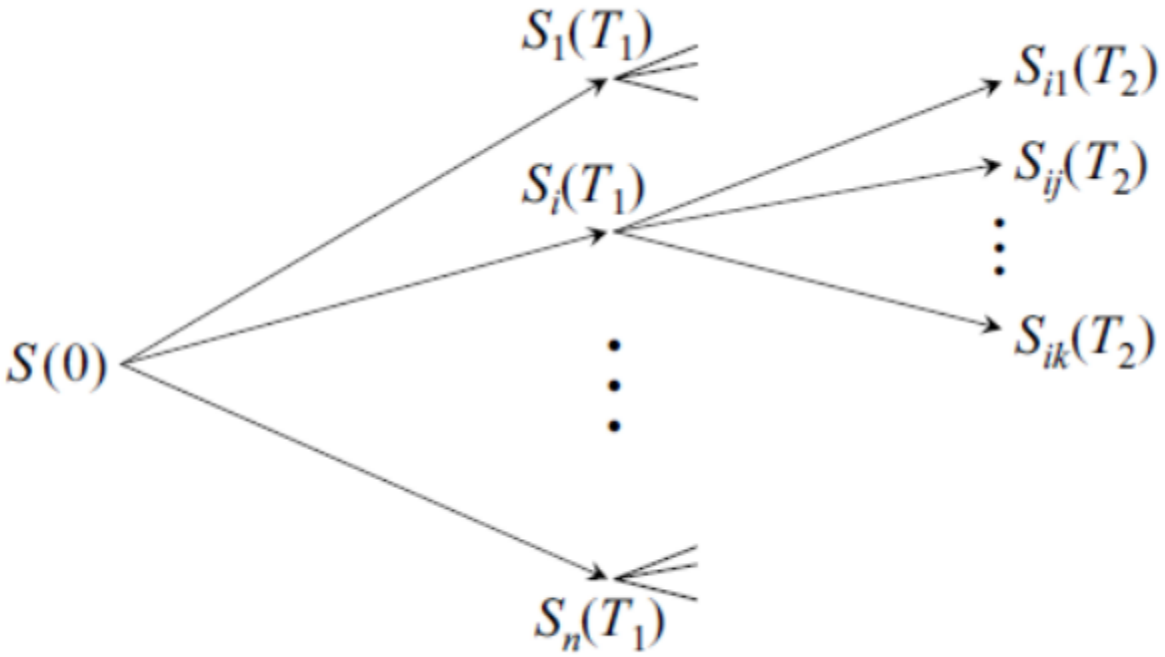
Bias / 3 example

Nonlinear functions of the mean value.

- If the dynamics of the underlying stock are described by the Black-Scholes model (i.e., $\frac{dS(t)}{S(t)} = rdt + \sigma dW(t)$), then $C^{(2)}$ and $C^{(1)}$ can be evaluated.
- Let's consider a situation where $C^{(1)}$ needs to be evaluated using simulations.
  - To achieve this, for an option with a term T_1, simulate n values $S_1(T_1), \dots, S_n(T_1)$ and subsequently k values $S_{i1}(T_2), \dots, S_{ik}(T_2)$ for the option with a term T_2, for each $S_i(T_1)$

Bias / 3 example

Nonlinear functions of the mean value.



Bias / 3 example

Nonlinear functions of the mean value.

- Evaluate the value of the “intrinsic” option at $S_i(T_1)$ using

$$\hat{C}_k^{(2)}\left(S_i(T_1)\right) = \frac{1}{k}\sum_{j=1}^{k} e^{-r(T_2-T_1)}\left(S_{ij}(T_2) - K_2\right)^+$$

- And then evaluate $C^{(1)}$ using

$$\hat{C}_n^{(1)} = \frac{1}{n}\sum_{i=1}^{n} e^{-rT_1}\left(\hat{C}_k^{(2)}\left(S_i(T_1)\right) - K_1\right)^+$$

Bias / 3 example

Nonlinear functions of the mean value.

- If the “Intrinsic” assessment $\hat{C}_k^{(2)}$ is replaced by its mathematical expectation, the result will yield an unbiased estimation for $C^{(1)}$.
- However, since the evaluation involves estimating the mathematical expectation of the “intrinsic” option, the overall estimation becomes biased.

This bias can be controlled by increasing the number of simulation iterations, but this also escalates computational costs.

Bias / 3 example

Nonlinear functions of the mean value.

In this example, the bias arose due to the application of a nonlinear function (option payoff) to estimate the mathematical expectation.

These types of errors occur in at least two critical Monte Carlo applications in financial engineering:

1. Assessing portfolio risk over a fixed period; The portfolio value at the end of the period is considered a conditional mathematical expectation.
2. Evaluating American-style options through simulation; The payoff of the option at each exercise date needs to be compared with the conditional expected discounted payoff.

Bias

- In all the discussed examples, the necessary estimation is sought as the average value of independent repetitions.
- Each repetition carries a bias, but each bias can be made sufficiently small by increasing the computational cost of repetitions.
- If there’s fixed computational budget, it’s essential to operate within that budget, dedicating more effort to minimize bias but reducing the number of repetitions that can be performed. Consequently, this could increase the variance of the estimation.

Variance

- It is necessary to find a way to measure the performance of an estimation that balances bias and variance.
- A standard measure for this is the Mean Squared Error (MSE):

$$MSE(\hat{a}) = E[(\hat{a} - a)^2] = Bias^2(\hat{a}) + Variance(\hat{a})$$

  where $\hat{a}$ is the estimation of quantity a.
- Often, calculating the precise Mean Squared Error (MSE) is impractical, but it is possible to compare estimations using their asymptotic MSE.

Variance

- For simplicity, let's focus on estimations that are sample means from i.i.d. repetitions.

$\hat{C}(n, \delta)$- the sample mean from n independent repetitions with parameter δ

δ – parameter that determines/characterizes bias

$$E\left[\hat{C}(n, \delta)\right] = a_\delta \ \ and \ \ a_\delta \to a, if \ \delta \to 0$$

- Let's consider the $MSE\left(\hat{C}(n, \delta)\right)$, as the computation budget increases.

Variance

- With additional conditions (typically, uniform integrability), applying the Central Limit Theorem asymptotically to the unbiased estimation $\hat{C}_{N(s)}$ yields:

$$sVar\left[\hat{C}_{N(s)}\right] \to \sigma_C^2 E[\tau]$$

$$\Longrightarrow s^{1/2} \sqrt{Var\left[\hat{C}_{N(s)}\right]} \to \sigma_C \sqrt{E[\tau]}$$

- s demonstrates the rate at which the standard error pf $\hat{C}_{N(s)}$ decreases.
- Similar information is obtained in the case of biased estimation.

Variance

Some assumptions about the estimation:

- τ_δ - calculation time for each repetition at parameter δ (δ is not a random variable)
- There exist constants for deviation in estimation and computation time $\eta, \beta > 0 \; and \; b, c > 0$, if $\delta \to 0$, then

$$a_\delta - a = b\delta^\beta + o(\delta^\beta)$$
$$\tau_\delta = c\delta^{-\eta} + o(\delta^{-\eta})$$

Variance

Given a computation budget s, we can specify the allocation of this budget to reduce bias and variance by setting the condition $s \rightarrow \delta(s)$ for a chosen parameter δ.

$N(s) = \lfloor s/\tau_{\delta(s)} \rfloor$ - resulting number of repetitions

$\hat{C}(s) \equiv \hat{C}\big(N(s), \delta(s)\big)$ - resulting estimation

Now the estimation has only one indicator – the parameter s.

Let's consider the distribution law $\delta(s)$, where

$$\delta(s) = as^{-\gamma} + o(s^{-\gamma})$$

Where constants $a, \gamma > 0$.

Variance

A larger γ corresponds to a smaller $\delta(s)$,

which means more effort is devoted to reducing bias.

$\Longrightarrow$ this leads to a smaller δ, increasing the computation time per iteration (as seen in Formula on slide 35);

$\Longrightarrow$ consequently, fewer iterations are performed, reducing the effort allocated to reducing variance.

- The goal is to relate the choice of γ to an indicator that reduces $MSE\left(\hat{C}(s)\right)$ as s increases.

Variance

- To analyze the MSE, the squared bias is expressed as

$$\left(a_{\delta(s)} - a\right)^2 = b^2\delta(s)^{2\beta} + o\left(\delta(s)^{2\beta}\right) =$$
$$= b^2 a^{2\beta} s^{-2\beta\gamma} + o\left(s^{-2\beta\gamma}\right) = O\left(s^{-2\beta\gamma}\right)$$

- Further analyzing the variance, denote the variance of repetition with parameter δ as σ_δ^2

$$Var\left(\hat{C}(s)\right) = \frac{\sigma_{\delta(s)}^2}{\left\lfloor s/\tau_{\delta(s)} \right\rfloor}$$

it is assumed that σ_δ^2 has a finite limit $\sigma^2 > 0$ as $\delta \to 0$.

Variance

- Under this condition, it holds true that

$$Var\left(\hat{C}(s)\right) = \frac{\sigma^2 \tau_{\delta(s)}}{s} + o\left(\tau_{\delta(s)}/s\right)$$

- Combining the previous equations with this one yields

$$Var\left(\hat{C}(s)\right) = \frac{\sigma^2 c\delta(s)^{-\eta}}{s} + o(\delta(s)^{-\eta}/s) =$$
$$= \sigma^2 c a^{-\eta} s^{\gamma\eta-1} + o(s^{\gamma\eta-1}) = O(s^{\gamma\eta-1})$$

Variance

- The effects of choosing different γ values:
  - If $2\beta\gamma > 1 - \gamma\eta$, then the distribution law ensures a faster convergence of the quadratic deviation to 0, compared to dispersion. Therefore, the Mean Squared Error (MSE) is more influenced by dispersion.
  - If $2\beta\gamma < 1 - \gamma\eta$, then for large s, the MSE is more affected by the quadratic deviation.
  - Optimal γ choice balances the impact of these two indicators.
  - Choosing $2\beta\gamma = 1 - \gamma\eta$ implies that $\gamma = 1/(2\beta + \eta)$. Substituting this value yields:

$$MSE\left(\hat{C}(s)\right) = \left(b^2 a^{2\beta} + \sigma^2 c a^{-\eta}\right) s^{-2\beta/(2\beta+\eta)} + o\left(s^{-2\beta/(2\beta+\eta)}\right)$$

Variance

- Thus, the square root of the Mean Squared Error is:

$$RMSE\left(\hat{C}(s)\right) \equiv \sqrt{MSE\left(\hat{C}(s)\right)} = O\left(s^{-2\beta/(2\beta+\eta)}\right)$$

- The power of s in this approximation indicates the convergence rate of RMSE, and it can be compared to the convergence rate of $s^{-1/2}$ in the case of an unbiased situation.

Variance

Conclusions:

- A large β corresponds to a rapid disappearance of bias; if $\beta \to \infty$, then $\beta/(2\beta + \eta) \to 1/2$, achieving the standard error convergence rate in the unbiased case.
- Similarly, if η is small, the computational costs for reducing bias are low; if $\eta \to 0$, then $\beta/(2\beta + \eta) \to 1/2$.
- However, choosing a finite β and positive η results in a slower convergence rate for an asymptotically unbiased estimation compared to an unbiased estimation.

Variance

- In Monte Carlo applications in financial engineering, the variance of an estimation is typically larger than the squared deviation.
- With a few notable exceptions (including pricing American options), it is easier to introduce simulations with sufficiently small biases than with sufficiently small standard errors.

Thank You!